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COMPUTABLE ELASTIC DISTANCES BETWEEN SHAPES*

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Abstract. We define distances between geometric curves by the square root of the minimal energy required to transform one curve into the other. The energy is formally defined from a left invariant Riemannian distance on an infinite dimensional group acting on the curves, which can be explicitly computed. The obtained distance boils down to a variational problem for which an optimal matching between the curves has to be computed. An analysis of the distance when the curves are polygonal leads to a numerical procedure for the solution of the variational problem, which can efficiently be implemented, as illustrated by experiments.

Key words. shape comparison, elastic matching, shape representation

AMS subject classifications. Primary, 68T10; Secondary, 53A04

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1. Introduction.

1.1. Generalities. The problem of matching two objects together is very important in computer vision and shape recognition. In many cases, recognition is based on shapes (outlines), with the help of some suitably designed distance. A general principle is to associate with any pair (O_1, O_2) of objects to be compared a measure of discrepancy $d(O_1, O_2)$. The recognition of an observed object O may be done by finding, from a dictionary of "templates," the previously recorded object O_{temp}, for which $d(O, O_{temp})$ is minimal. Clearly, the definition of the distance is the crucial step of the method, and much research has been done in this direction. We shall not try here to provide a review of the huge literature existing on the subject (see, for example, [17]) but rather focus on methods related to deformable templates, with which we are directly concerned.

Instead of basing recognition on a finite collection of points of interest (primitives) taken from the outline of an object (corners, inflexion points, etc.), which is a popular way of handling the problem, our purpose is to base the comparison on the whole outline, considered as a plane curve. The distance we shall define incorporates some deformation energy between the curves. The approach, as we will see, turns out to be intrinsic and robust to usual Euclidean transformations.

The method is related to the wide literature on "snakes" [14], [7], [21] etc. in the way that our distance corresponds to some continuous process of deformation of one curve into another. It is also related to papers on elastic matching, such as [8]; however, we provide an elastic matching algorithm which is based on a true distance between intrinsic properties of the shapes, taking into account possible invariance to scaling or Euclidean transformations in the case they are required. From this point of view, our results are indebted to the seminal work of Grenander on group theory applied to pattern recognition (cf. [10] and [11], in particular; see also [1], [2], 12).

Another source of inspiration may come from mathematical physics, since we are going to look to the path (process) of lowest energy which deforms one object into

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another, as studied, for example, in fluid mechanics (least action principle; cf. [19], [5]). A good general formulation may be found in the appendix of [3].

1.2. Principles of the approach. We denote by $\mathcal{C}$ our object space. Assuming that each object in $\mathcal{C}$ can in some way be deformed into another, our purpose is to define a distance measuring the amount of deformation which is necessary for this operation. The deformations are represented by a *group action*

$$\begin{aligned} G \times \mathcal{C} &\longrightarrow \mathcal{C}, \\ (a, C) &\longrightarrow a.C \end{aligned}$$

on $\mathcal{C}$, where G is a group (the fact that we have a group action means that, for all $a, b \in G$ and for all $C \in \mathcal{C}$, $a.(b.C) = (ab).C$ and $e.C = C$, e being the identity element of G). The fact that each object can be deformed into another means that the group action is transitive; that is, we assume that, for all C_1, C_2 in $\mathcal{C}$, there exists a in G such that $a.C_1 = C_2$.

Assume that we have some way of measuring the cost of the transformation $C \to a.C$, and denote this cost by $\Gamma(a, C)$. To compare two objects, we put

$$d(C_1, C_2) = \inf\{\Gamma(a, C_1), C_2 = a.C_1\}, \tag{1}$$

which is the smaller cost required to deform C_1 into C_2. The following proposition then is almost obvious.

PROPOSITION 1. *Assume that G acts transitively on $\mathcal{C}$ and that Γ is a function defined on $G \times \mathcal{C}$, taking values in $[0, +\infty[$, such that*

i) *for all $C \in \mathcal{C}$, $\Gamma(e, C) = 0$,*

ii) *for all $a \in G$, $C \in \mathcal{C}$, $\Gamma(a, C) = \Gamma(a^{-1}, a.C)$,*

iii) *for all $a, b \in G$, $C \in \mathcal{C}$, $\Gamma(ab, C) \leq \Gamma(b, C) + \Gamma(a, bC)$.*

Then d defined by (1) *is symmetric, satisfies the triangle inequality, and is such that $d(C, C) = 0$ for all $C \in \mathcal{C}$.*

When G acts transitively on $\mathcal{C}$, we know from elementary group theory that $\mathcal{C}$ can be identified (at least as a set) to a coset space on G. Indeed, fixing some reference element $C_0 \in \mathcal{C}$ and, letting $H_0 = \{h \in G, h.C_0 = C_0\}$, $\mathcal{C}$ can be identified to G/H_0 through the well-defined correspondence $a.H_0 \leftrightarrow a.C_0$. Using this identification, the following proposition provides a way to define a suitable cost function Γ provided that G itself is equipped with a left-invariant distance.

PROPOSITION 2. *Let d_G be a distance on G. Assume that there exists $\gamma_0 : G \to \mathbb{R}$ such that $\gamma_0(h) = 1$ if $h \in H_0$ and, for all $a, b, c \in G$,*

$$d_G(ca, cb) = \gamma(c) d_G(a, b). \tag{2}$$

Define for $C \in \mathcal{C}$, with $C = b.C_0$,

$$\Gamma_0(a, C) = d_G(e, a^{-1})/\gamma(b)\,. \tag{3}$$

Then Γ_0 satisfies properties i), ii), *and* iii) *of Proposition* 1. *The obtained distance is, if $C = b.C_0$,*

$$d_0(C, C') = \gamma(b)^{-1} \inf\{d(e, a), aC' = C\}\,.$$

Proof. Here again the proof is almost obvious. Note first that γ must satisfy $\gamma(ab) = \gamma(a)\gamma(b)$. Γ_0 is well defined, since $C = b.C_0 = \tilde{b}.C_0$ implies that $b^{-1}\tilde{b} \in H$

so that $\gamma(b) = \gamma(\tilde{b})$. Let us check, for example, property iii), leaving i) and ii) to the reader. We have, if $C = b.C_0$,

$$\begin{aligned}\Gamma_0(aa', C) = \gamma(b)^{-1} d_G(e, (aa')^{-1}) &\le \gamma(b)^{-1}[d_G(e, (a')^{-1}) + d_G((a')^{-1}, (aa')^{-1})] \\ &= \Gamma_0(a', C) + \gamma(a'b)^{-1} d(e, a^{-1}) = \Gamma_0(a', C) + \Gamma_0(a, a'C)\,.\end{aligned}$$

Finally, we have

$$d_0(C, C') = \gamma(b)^{-1} \inf\{d(e, a^{-1}), C' = a.C\} = \gamma(b)^{-1} \inf\{d(e, a), a.C' = C\}\,. \qquad \square$$

If we had chosen another reference object C_1 instead of C_0, yielding another identification of $\mathcal{C}$ (by G/H_1), we have, if $C_1 = a_1 C_0$, $H_1 = a_1 H_0 a_1^{-1}$ so that $\gamma \equiv 1$ also on H_1 and the obtained cost function is, if $C = b.C_1$,

$$\Gamma_1(a, C) = d(e, a^{-1})/\gamma(b) = \gamma(a_1)\Gamma_0(a, C)\,,$$

so that the cost function is canonical up to a multiplicative factor.

The problem is then to define a suitable left-invariance distance on G. Our intuition will come from standard constructions of differential geometry. We think of G as a Lie group acting on $\mathcal{O}$, on which we define an invariant metric characterized by its trace on the Lie algebra of G. Our objects being plane curves, the group G (which must act transitively on $\mathcal{C}$) must be infinite dimensional. In fact, it will appear that a natural way to define it will be as a quotient space of a dense subset of an infinite dimensional Hilbertian manifold (either a Hilbert space or a Hilbert sphere) so that G itself will not even be a manifold. For these reasons, and because we want to keep this paper as elementary as possible, we will refrain from speaking with differential geometric terms, but rather use the following kind of argument.

To define a geodesic distance on G, we need only to know how to compute the lengths of smooth paths in G. If $\mathbf{a} = (\mathbf{a}(t), t \in [0,1])$ is such a path subject to suitable regularity conditions which will depend on the context, we must define the length $L(\mathbf{a})$ and then set

$$d_G(a_0, a_1) = \inf\{L(\mathbf{a}), \mathbf{a}(0) = a_0, \mathbf{a}(1) = a_1\},$$

the infimum being computed over some set of *admissible paths.* As soon as the following hold:

a) if $\mathbf{a}(t), t \in [0,1]$ is admissible, so is $\mathbf{a}(1-t), t \in [0,1]$ and both paths have the same length, and

b) if $\mathbf{a}(.)$ and $\tilde{\mathbf{a}}(.)$ are admissible, so is their concatenation, equal to $\mathbf{a}(2t)$ for $t \in [0, 1/2]$ and to $\tilde{\mathbf{a}}(2t-1)$ for $t \in [1/2, 1]$, and the length of the concatenation is the sum of the lengths of $\mathbf{O}$ and $\tilde{\mathbf{O}}$,

the function d_G is symmetrical and satisfies the triangle inequality.

We would like to define the length of a path $t \to \mathbf{a}(t)$ by the formula

$$\int_0^1 \|\dot{\mathbf{a}}_t(t)\| dt \tag{4}$$

for some norm. Thinking of $\dot{\mathbf{a}}_t(t)dt$ as a way to represent the portion of path between $\mathbf{a}(t)$ and $\mathbf{a}(t+dt)$, defining the norm corresponds to defining the cost of a small variation of $a(t)$. Note that we must have

$$d_G(\mathbf{a}(t), \mathbf{a}(t+dt)) = \gamma(\mathbf{a}(t)) d_G(e, \mathbf{a}(t)^{-1}\mathbf{a}(t+dt))$$

so that the problem is to define γ and the cost of a small variation from the identity.

For this purpose, we will consider some small deformation of a plane curve C into some plane curve $C + \delta C$. This deformation will easily be interpreted, and we shall choose some natural way to define its cost Γ. We shall then try to represent $C \to C + \delta C$ as a transformation $C \to a.C$ for some group action on $\mathcal{C}$ and identify the cost Γ under the form $\Gamma = \tilde{\gamma}(C)^{-1} d_G(e, a^{-1})$ at least for a close to identity. Now, having chosen a reference curve C_0 and expressed $C = b.C_0$, we see that we have to set $\gamma(b) = \tilde{\gamma}(C)$.

2. Comparison of plane curves.

2.1. Infinitesimal deformations. Consider a plane curve in parametric form

$$C = \{m(s) = (x(s), y(s)),\ s \in [0, l]\}.$$

We assume that the parametrization is done by arc-length; that is (denoting by $\dot{f}_s$ the derivative of a function f with respect to s): $\dot{x}_s^2 + \dot{y}_s^2 \equiv 1$, so that l is the Euclidean length of the curve C. This is the only assumption which will be done concerning the regularity of the curves which are compared.

We first justify, heuristically, the introduction of the group G and compute the cost of an infinitesimal deformation of C. To start, assume that by some deformation each point of C is moved, the displacement being given by some vector field carried by C (i.e., a function $V(s) = (u(s), v(s))$, considered as a vector starting at the point $m(s)$). A new curve is obtained, which is

$$\tilde{C} = \{\tilde{m}(s) = (x(s) + u(s), y(s) + v(s))\}.$$

The field V (and its derivatives) is infinitely small.

- We define the *energy* of this deformation by

$$\delta E^{(3)}(V) = \int_0^l \|\dot{V}_s(s)\|^2 ds$$

and its cost by the square root of this energy. This cost is null if $\tilde{C}$ is a translation of C, since in this case one may take V constant (we are in fact viewing curves modulo translations). It is also rotation invariant and weakly scale invariant: if C and V are simultaneously rotated and scaled by a common factor λ, the cost of the deformation is scaled by $\sqrt{\lambda}$. We shall define in the sequel infinitesimal distances having more robustness with respect to these operations.

We now see how the decomposition of $\dot{V}_s$ on the tangential and normal direction of C at $m(s)$ can help to define a group action on plane curves. For this, denote by g^* the function $g^* : [0, l] \to [0, \tilde{l}]$, which associates with s the arc length $\tilde{s}$ in $\tilde{C}$ of the point $m(s) + V(s)$. At order 1, we have

$$\dot{g}_s^* = 1 + \dot{u}_s \dot{x}_s + \dot{v}_s \dot{y}_s. \tag{5}$$

In other terms, $\dot{g}_s^* - 1$ is the tangential part of $\dot{V}_s$

Moreover, denote by $\theta^*(s)$ the angle between the tangent to C at point $m(s)$ and the axis $y = 0$. Let $\tilde{\theta}^*(\tilde{s})$ be the similar function defined for $\tilde{C}$. We have

$$\begin{cases} \cos\theta^* = \dot{x}_s, \\ \sin\theta^* = \dot{y}_s \end{cases}$$

and (still at order 1)

$$\begin{cases} \cos\tilde{\theta}^* \circ g^* = (\dot{x}_s + \dot{u}_s)(1 - \dot{u}_s\dot{x}_s - \dot{v}_s\dot{y}_s) \simeq \dot{x}_s - \dot{y}_s(-\dot{y}_s\dot{u}_s + \dot{x}_s\dot{v}_s), \\ \sin\tilde{\theta}^* \circ g^* = (\dot{y}_s + \dot{v}_s)(1 - \dot{u}_s\dot{x}_s - \dot{v}_s\dot{y}_s) \simeq \dot{y}_s + \dot{x}_s(-\dot{y}_s\dot{u}_s + \dot{x}_s\dot{v}_s). \end{cases}$$

Let $D^* = -\dot{y}_s\dot{u}_s + \dot{x}_s\dot{v}_s$; it is the normal component of $\dot{V}_s$. At order 1, $\sin D^* = D^*$, $\cos D^* = 1$, and we may write

$$\begin{cases} \cos\tilde{\theta}^* \circ g^* \simeq \cos(\theta^* + D^*), \\ \sin\tilde{\theta}^* \circ g^* \simeq \sin(\theta^* + D^*), \end{cases}$$

hence,

$$\tilde{\theta}^* \circ g^* - \theta^* = D^* = -\dot{y}_s\dot{u}_s + \dot{x}_s\dot{v}_s\,; \tag{6}$$

this equality is true modulo 2π. We can use the version of the left-hand term, which is infinitely small of order 1, to induce a true equality. We therefore obtain another expression for $\delta E^{(3)}$, which is

$$\delta E^{(3)} = \int_0^l (\dot{g}_s^* - 1)^2 ds + \int_0^l (\tilde{\theta}^* \circ g^*(s) - \theta^*(s))^2 ds.$$

Note that g^* and D^* implicitly refer to l, the length of C, since both functions are defined on $[0, l]$. We shall set $g(s) = g^*(ls)/\tilde{l}$, which is defined on $[0, 1]$ and takes values in $[0, 1]$. We let $\lambda = \tilde{l}/l$. We also let $\theta(s) = \theta^*(ls)$ and $\tilde{\theta}(\tilde{s}) = \tilde{\theta}^*(\tilde{l}\tilde{s})$, which also are defined on $[0, 1]$. We have $\dot{g}_s(s) = \lambda^{-1}\dot{g}_s^*(ls)$ and $\tilde{\theta}^* \circ g^*(ls) = \tilde{\theta} \circ g(s)$, so that

$$\delta E^{(3)} = l\int_0^1 (\lambda\dot{g}_s(s) - 1)^2 ds + l\int_0^1 (\tilde{\theta} \circ g(s) - \theta(s))^2 ds\,.$$

Letting $D(s) = \tilde{\theta} \circ g(s) - \theta(s)$, $\delta E^{(3)}$ may be seen as a function of λ, g, D and l. We shall write

$$\delta E^{(3)}(\lambda, g, D, l) = l\int_0^1 \left[(\lambda\dot{g}_s - 1)^2 + D^2\right] ds.$$

Finally, still taking terms of first order, this may be rewritten

$$\delta E^{(3)}(\lambda, g, D, l) = l(\lambda - 1)^2 + l\int_0^1 \left[(\dot{g}_s - 1)^2 + D^2\right] ds.$$

We now see how the functional $\delta E^{(3)}$ involves some action on the curve C. First we note that the pair $(l, \theta(.))$ characterizes a curve C up to translations. Since $\theta(.)$ is defined modulo 2π, we make the identification of a curve C (modulo translations) and a pair $(l, \zeta(.))$ in which $l \in]0, +\infty[$ and ζ is a function defined on $[0, 1]$ with values in the unit circle of $\mathbb{C}$. The relations between C and (l, ζ) is that l is the length of C, and $C = \{(x(s), y(s)), s \in [0, l]\}$, where s is the arc-length parametrization of C and $\dot{x}_s = \Re(\zeta)$ and $\dot{y}_s = \Im(\zeta)$ (the real and imaginary parts of ζ). From this remark, we represent our set of objects as

$$\mathcal{C} = \{(l, \zeta), l > 0, \zeta : [0, 1] \to \Gamma_1, \text{measurable}\}, \tag{7}$$

where Γ_1 is the unit circle of $\mathbb{C}$. Note that in this representation, ζ is translation and scale invariant, and the effect of a rotation on $C = (l, \zeta)$ corresponds to a multiplication of ζ by a constant complex number of modulus 1.

In view of the computations above, the transformation which can naturally be associated to λ, g, and D is

$$(l, \theta) \to (l/\lambda, \theta \circ g^{-1} + D \circ g^{-1}) = (\tilde{l}, \tilde{\theta})\,. \tag{8}$$

Now, define the action

$$(\lambda, g, r).(l, \zeta) = (l/\lambda, r.\zeta \circ g), \tag{9}$$

where $\lambda > 0$, g is a diffeomorphism of $[0, 1]$ and r is a measurable function, defined on $[0, 1]$, with values in Γ_1. Let G be the set composed with these 3-uples (we shall give a precise definition of G below). Let us define the product $(\tilde{\lambda}, \tilde{g}, \tilde{r}).(\lambda, g, r)$ so that

$$(\tilde{\lambda}, \tilde{g}, \tilde{r})[(\lambda, g, r).(l, \zeta)] = [(\tilde{\lambda}, \tilde{g}, \tilde{r})(\lambda, g, r)].(l, \zeta)\,,$$

which yields

$$(\tilde{\lambda}, \tilde{g}, \tilde{r}).(\lambda, g, r) = (\lambda\tilde{\lambda}, g \circ \tilde{g}, \tilde{r}.r \circ \tilde{g})\,. \tag{10}$$

This is a group operation, with identity $e_G = (1, \mathrm{Id}, \mathbf{1})$ (where $\mathrm{Id}(s) = s$ and $\mathbf{1}(s) = 1$), and inverse $(\lambda, g, r)^{-1} = (1/\lambda, g^{-1}, \overline{r} \circ g^{-1})$, where $\overline{r}$ is the complex conjugate of r ($\overline{r}.r \equiv 1$). Now, letting $r = e^{-iD}$, the relation (9) may be rewritten

$$(l, \theta) \to (\lambda, g, r)^{-1}.(l, \theta)\,. \tag{11}$$

Our evaluation of small deformations implies that if (λ, g, r) is close to $(1, \mathrm{Id}, \mathbf{1})$, the effect of the action on (l, θ) is

$$\delta E^{(3)}(\lambda, g, r, l) = l(\lambda - 1)^2 + l \int_0^1 \left[(\dot{g}_s - 1)^2 + |r - 1|^2\right] ds\,, \tag{12}$$

with the first order approximation $|e^{-iD} - 1| \simeq |D|$.

Letting $a = (\lambda, g, r)$, we thus have evaluated the cost of a small deformation $C \to a^{-1}.C$ by

$$\Gamma(a^{-1}, C)^2 = l(\lambda - 1)^2 + l \int_0^1 \left[(\dot{g}_s - 1)^2 + |r - 1|^2\right] ds\,.$$

Let us choose a reference curve C_0 as the horizontal plane segment of length 1; that is, $C_0 = (1, \mathbf{1})$. The curve $C = (l, \zeta)$ is equal to $b.C_0$ if and only if $b = (\lambda, g, r)$ with $\lambda = 1/l$ and $r = \zeta$. This above expression can be written

$$\Gamma(a^{-1}, C)^2 = d_G(e, a)^2/\gamma(b)^2$$

with

$$\gamma(b) = 1/\sqrt{l} \tag{13}$$

and

$$d_G(e, a)^2 = (\lambda - 1)^2 + \int_0^1 \left[(\dot{g}_s - 1)^2 + |r - 1|^2\right] ds\,. \tag{14}$$

We thus have defined our function γ and the distance between e and an element $a \in G$ infinitely close to e. Note that $\gamma(bb') = \gamma(b)\gamma(b')$ and $\gamma(b) = 1$ if $b.C_0 = C_0$.

At this point, to extend the distance to arbitrary deformations, we have the possibility to carry on by putting on G a structure of infinite dimensional Banach manifold (restricting, for example, to C^1 diffeomorphisms g), adding a Riemannian structure consistently with our discussion, and computing the associated geodesic distance (cf. [9], [16]). It will, however, be simpler and quicker to work in another way, identifying G to some subset of a Hilbert space, which is the way we follow now (see also [20] for a rigorous general construction of a distance in a similar context).

2.2. Rigorous definition of G. We now give a precise definition of G, which may seem rather awkward but will lead to very easy arguments and computations. We start by considering the Hilbert space $\mathcal{L}^2 = L^2([0,1], \mathbb{C})$ composed with all measurable functions X, defined on $[0,1]$, with values in $\mathbb{C}$, such that

$$\|X\|_2^2 := \int_0^1 |X(s)|^2 ds < \infty .$$

For $X \in \mathcal{L}^2$, we define

$$g^X(s) = \int_0^s |X(s)|^2 ds / \int_0^1 |X(s)|^2 ds , \tag{15}$$

and consider the product in $\mathcal{L}^2$:

$$(X \star Y)(s) = X(s)\, Y \circ g^X(s) . \tag{16}$$

For this product to be well defined, the result must not depend on the class of Y modulo the sets of null measure, and this is equivalent to the condition that $|X| > 0$ almost everywhere. Thus, let

$$\tilde{G} = \{X \in \mathcal{L}^2, |X| > 0 \text{ almost everywhere}\}. \tag{17}$$

We have the following proposition.

PROPOSITION 3. *$\tilde{G}$ is a group for the operation $\star$.*

Proof. We first prove that if $X, Y \in \tilde{G}$, then $X \star Y \in \tilde{G}$, and so compute $\int_0^1 |X(s)\, Y \circ g^X(s)|^2 ds$. We make the change of variable $u = g^X(s)$ to get

$$\int_0^1 |X \star Y|^2 ds = \int_0^1 |X|^2 ds \int_0^1 |Y|^2 ds , \tag{18}$$

(the change of variables is valid since $|X| > 0$ almost everywhere; see [13, corollary (20.5)], for example).

The fact that $\star$ is associative may be proved by elementary arguments.

The function $\mathbf{1}$ clearly is an identity for $\star$ and belongs to $\tilde{G}$. Assume that $X \in \tilde{G}$; g^X is then strictly increasing and its inverse h is well defined and strictly increasing. However, with the change of variable $v = g^X(u)$, we have

$$\int_0^s |X \circ h(v)|^{-2} dv = \int_0^{h(s)} |X(u)|^{-2} \left[|X(u)|^2 / \int_0^1 |X(v)|^2 dv \right] du;$$

that is, $\int_0^s |X \circ h(v)|^{-2} dv = h(s) / \int_0^1 |X(v)|^2 dv$. This implies that $Y := 1/(X \circ h) \in \mathcal{L}^2$, that

$$\int_0^s |Y(v)|^2 dv = h(s) / \int_0^1 |X(u)|^2 du ,$$

and thus that $h = g^Y$. We have $X \star Y = Y \star X = \mathbf{1}$, and $Y \in \tilde{G}$. Thus, Y is the inverse of X and the proof of Proposition 3 is completed. □

We now put, for $X \in \tilde{G}$, $\lambda^X = \int_0^1 |X|^2 du$, $r^X = X^2/|X|^2$, which is defined almost everywhere, and denote by Φ the mapping

$$\Phi : X \to (\lambda^X, g^X, r^X) .$$

We shall *define* our group G acting on plane curves to be the image of $\tilde{G}$ by Φ, which is the following definition.

DEFINITION 1. *We denote by G the set of 3-uples (λ, g, r) subject to the conditions*

- $\lambda \in]0, +\infty[$,
- *g is continuous on $[0,1]$, with values in $\mathbb{R}$, and is such that*
  - $g(0) = 0$, $g(1) = 1$;
  - *there exists a function $q > 0$ almost everywhere on $[0,1]$ such that*

$$g(s) = \int_0^s q^2(\sigma) d\sigma;$$

- *r is measurable, $r : [0,1] \to \Gamma_1$, where Γ_1 is the unit circle in $\mathbb{C}$.*

The product (10) is well defined on G, and the first basic remark is Proposition 4.

PROPOSITION 4. *$\Phi : \tilde{G} \to G$ is a group homomorphism.*

Proof. These are straightforward computations. We have $\lambda^{X\star Y} = \lambda^X \lambda^Y$ by (18). We also have

$$\lambda^{X\star Y} g^{X\star Y}(s) = \int_0^s |X|^2 |Y \circ g^X|^2 du = \lambda^X \int_0^{g^X(s)} |Y(u)|^2 du = \lambda^X \lambda^Y g^Y \circ g^X(s).$$

Thus, $g^{X\star Y} = g^Y \circ g^X(s)$ and the fact that $r^{X\star Y} = r^X r^Y \circ g^X$ is obvious. $\square$

Note that Φ is not one to one: we have $\Phi(X) = \Phi(Y)$ if and only if $X^2 = Y^2$ almost everywhere. Denoting by $\mathcal{R}$ the equivalence relation $X^2 = Y^2$, we can identify G with the quotient space $\tilde{G}/\mathcal{R}$.

We therefore have an identification between G and $\tilde{G}/\mathcal{R}$, and the crucial remark is that the usual L^2-norm on $\tilde{G}$ (up to a scalar factor) is the correct counterpart of the distance $d_G(e,a)$ which has been obtained for $a \simeq e$ from heuristic considerations; that is, our identification is (in an informal sense) isometric in the neighborhood of the identity. Thus, let us check how the L^2 norm on $\mathcal{L}^2$ is consistent with formula (14). For this, consider a small perturbation of $\mathbf{1}$ in $\mathcal{L}^2$, of the kind $Y = \mathbf{1} + X$, and assume that $|X(s)|$ is small for all s. We have

$$\lambda^Y - 1 \simeq 2 \int_0^1 \Re(X),$$

$$\dot{g}_s^Y - 1 = |Y|^2/\lambda^Y - 1 \simeq 2\Re(X) - 2\int_0^1 \Re(X),$$

and

$$r^Y - 1 = Y^2/|Y|^2 - 1 \simeq 2\Im(X)$$

so that

$$(\lambda^Y - 1)^2 + \int_0^1 \left[(\dot{g}_s^Y - 1)^2 + |r^Y - 1|^2\right] \simeq 4\int_0^1 |X|^2;$$

that is, we retrieve formula (14) up to a factor 4. Thus, $d_G(e, \Phi(Y))$ is identified for $Y \simeq \mathbf{1}$ to $2\|Y - \mathbf{1}\|_2$. Note also that if $\Phi(\tilde{Y}) = \Phi(Y)$ and $Y(s)$ is close to 1 for all s, necessarily $\|\tilde{Y} - \mathbf{1}\|_2 \geq \|Y - \mathbf{1}\|_2$, so that $d_G(e,a)$ for a close to e is the infimum of $2\|Y - \mathbf{1}\|$ over all Y such that $\Phi(Y) = a$, which is the quotient distance on $\tilde{G}/\mathcal{R}$.

Now, we check left translation compatibility in order to translate the above remark to the neighborhood of any point $a \in G$. If $X \in \tilde{G}$, let $T_X : Y \to X \star Y$ be the left translation on $\tilde{G}$; T_X is linear and can be extended to all $Y \in \mathcal{L}^2$. However, we have for $X \in \tilde{G}$, $Y \in \mathcal{L}^2$,

$$\|T_X Y\|_2^2 = \int_0^1 |X|^2 |Y \circ g^X|^2 ds = \lambda^X \int_0^1 |Y|^2 ds = \lambda^X \|Y\|_2^2 .$$

This means that if $a = \Phi(X)$ and $b = \Phi(Y)$ with $X \simeq Y$, we have, from (13), $\gamma(a) = 1/\sqrt{\lambda^X}$ and

$$d_G(a,b) = \gamma(a) d_G(e, a^{-1}b) = (2/\sqrt{\lambda^X}) \|\mathbf{1} - X^{-1} \star Y\|_2 = 2\|X - Y\|_2 .$$

Thus, the norm of a small variation $\Phi(X) \to \Phi(Y)$ in G is given by $2\|X - Y\|_2$. This will enable us to easily define the lengths of a path in G from the length of corresponding paths in $\tilde{G}$.

2.3. Admissible paths in G. We therefore have a representation of our problem within a Hilbert space context. We can use this representation to define admissible paths on which lengths can be computed. We start with admissible paths in $\mathcal{L}^2$

DEFINITION 2. *A path $(\mathbf{X}(t,.), t \in [0,1])$ is said to be admissible in $\mathcal{L}^2$ $(\mathbf{X}(t,.) \in \mathcal{L}^2$ for all $t)$ if there exists a path, denoted $(\dot{\mathbf{X}}_t(t,.), t \in [0,1])$, such that*

- *for all $\phi \in \mathcal{L}^2$, the scalar function*

$$t \to \int_0^1 \mathbf{X}(t,s)\overline{\phi}(s)ds$$

is differentiable in the generalized sense [6], *and its derivative is*

$$t \to \int_0^1 \dot{\mathbf{X}}_t(t,s)\overline{\phi}(s)ds.$$

- *The total energy is finite:*

$$\int_0^1 \int_0^1 |\dot{\mathbf{X}}_t(t,s)|^2 dtds < \infty.$$

$(\mathbf{X}(t,.), t \in [0,1])$ *is admissible in $\tilde{G}$ if it is admissible in $\mathcal{L}^2$ and $(s \to \mathbf{X}(t,s)) \in \tilde{G}$ for all t.*

The length of an admissible path in $\mathcal{L}^2$ is

$$\tilde{L}(\mathbf{X}) = \int_0^1 \left[\int_0^1 |\dot{\mathbf{X}}_t(t,s)|^2 ds \right]^{1/2} dt. \tag{19}$$

This definition obviously satisfies the natural properties with respect to time reversal and concatenation.

Passing to G, we have the following definition.

DEFINITION 3. *A path $\mathbf{a}(t), t \in [0,1]$ is admissible in G if and only if there exists a path $\mathbf{X}(t,.), t \in [0,1]$ which is admissible in $\tilde{G}$ and such that, for all t, $\Phi(\mathbf{X}(t,.)) = \mathbf{a}(t)$. We now define the length of a path $\mathbf{a}$ in G acting on $C = (l, \theta)$ (denoted $L_l[\mathbf{a}]$) as $2\sqrt{l}$ times the length of a corresponding path in $\tilde{G}$ such that $\Phi(\mathbf{X}(t,.)) = \mathbf{a}(t)$. Because of the following proposition, there is no ambiguity.*

PROPOSITION 5. *If two admissible paths in $\mathcal{L}^2$, $\mathbf{X}(t,.)$ and $\mathbf{Y}(t,.)$ satisfy*

$$\mathbf{X}(t,.)^2 = \mathbf{Y}(t,.)^2$$

for all t, then

$$\dot{\mathbf{X}}_t(t,.)^2 = \dot{\mathbf{Y}}_t(t,.)^2 .$$

This proposition is proved in section 4.

2.4. Invariant distance associated with G. We now can compute explicitly the distance between two elements a and b in G as the length of the shortest admissible path in G joining them. According to the previous paragraph, this is the minimum among the lengths of the shortest paths in $\tilde{G}$ joining any X and Y such that $\Phi(X) = a$ and $\Phi(Y) = b$.

Paths of shortest length in $\mathcal{L}$ are straight lines, but, if $X, Y \in \tilde{G}$, the straight line $t \to tX + (1-t)Y$ does not necessarily stay within $\tilde{G}$; however, the length of this straight line is $\|X - Y\|_2$, and we always have

$$d_G(a,b) \geq 2 \min\{\|X - Y\|_2, X, Y \in \tilde{G}, \Phi(X) = a, \Phi(Y) = b\} . \tag{20}$$

Equality will be true provided that we show that the minimum in the right-hand term is attained for some X, Y such that $t \to tX + (1-t)Y$ stays within $\tilde{G}$; however, since $\|X - Y\|_2^2 = \int_0^1 (|X|^2 + |Y|^2 - 2\int_0^1 \Re(X\overline{Y})) = \lambda^X + \lambda^Y - 2\int_0^1 \Re(X\overline{Y})$, and since the signs of X and Y are arbitrary, the minimum in (20) is attained for X and Y with $\Re(X\overline{Y}) \geq 0$ everywhere. This implies, however, that $|tX + (1-t)Y|^2 \geq t^2|X|^2 + (1-t)^2|Y|^2$ and is thus positive almost everywhere. Putting everything together, we obtain the following theorem.

THEOREM 1. *One defines a distance on G by (for $a = (\lambda, g, e^{i\Delta})$, $b = (\mu, h, e^{i\tilde{\Delta}})$)*

$$d_G^{(3)}(a,b) = 2\left(\lambda + \mu - 2\sqrt{\lambda\mu}\int_0^1 \sqrt{\dot{g}_s \dot{h}_s}\left|\cos\left(\frac{\Delta - \tilde{\Delta}}{2}\right)\right| ds\right)^{1/2} . \tag{21}$$

One defines a distance between two plane curves $C = (l, e^{i\theta})$ and $\tilde{C} = (\tilde{l}, e^{i\tilde{\theta}})$,

$$d^{(3)}(C, \tilde{C}) = \left(l + \tilde{l} - 2\sqrt{l\tilde{l}}\sup_g \int_0^1 \sqrt{\dot{g}_s(s)}\left|\cos\frac{\tilde{\theta}\circ g(s) - \theta(s)}{2}\right| ds\right)^{1/2} , \tag{22}$$

the supremum being taken over functions g which are increasing diffeomorphisms of $[0,1]$.

Proof. We have obtained $d_G^{(3)}$ by computing $\|X - Y\|$ for $\Phi(X) = a$, $\Phi(Y) = b$ and $\Re(X\overline{Y}) \geq 0$.

If $C = (l, \zeta)$, we have $C = b.C_0$ with $b = (l, \mathrm{Id}, \zeta)$, and we thus have $\gamma(b) = 1/\sqrt{(l)}$. By Proposition 2, we have

$$d^{(3)}(C, \tilde{C}) = \sqrt{l}\inf\{d^{(3)}(e, a), a.\tilde{C} = C\},$$

but $a.(\tilde{l}, e^{i\tilde{\theta}}) = (l, e^{i\theta})$ if and only if $a = (\tilde{l}/l, g, e^{i\theta - i\tilde{\theta}\circ g})$ (cf. equation (8), which yields formula (22)).

It therefore only remains to prove Lemma 1.

LEMMA 1. *One has*

$$d^3(C, \tilde{C}) = 0 \Rightarrow l = \tilde{l} \text{ and } \theta = \tilde{\theta} .$$

This is proved in section 4. $\square$

2.5. Remark. We can also interpret G as the set of *parametric* curves; that is, one can associate with an element $(\lambda, g, r) \in G$ the parametric curve

$$\begin{aligned} C_{\lambda,g,r} &: [0,1] \to \mathbb{R}^2 \simeq \mathbb{C} \\ &u \to 1/\lambda \int_0^{g(u)} r(v)dv. \end{aligned}$$

We first see that an element of $\mathcal{C}$ is identified to the set of all parametric curves modulo a change of parameter, that is, to a geometric curve, as could have been expected. We can rewrite the left action of G on itself as

$$\begin{aligned} (\lambda, g, r).C_{l,\psi,\zeta} &: [0,1] \to \mathbb{C}, \\ &u \to (1/l\lambda) \int_0^{\psi \circ g(u)} r(v)\zeta \circ g(v)dv. \end{aligned}$$

Let us give an interpretation of this formula, which may be a little more intuitive. Assume that it is possible to program a car so that it will follow a certain path with given variation of speed without any operation of the driver. Assume moreover that there exist robot-drivers which can be programmed to drive a car in order to follow another path with another kind of speed variation. Let the programmed robot drive the programmed car so that both commands are combined: the obtained path and speed variation are the products of the first two as we have defined it.

3. Definition of distances with invariance restrictions. We now modify the previous distance by requiring some additional Euclidean invariance properties. Let us fix some terminology. A distance d on $\mathcal{C}$ is said to be *invariant* by a group of transformations Σ acting on $\mathcal{C}$ if, for all $\sigma \in \Sigma$, for all C_1, C_2, we have

$$d(\sigma C_1, \sigma C_2) = d(C_1, C_2).$$

The distance is said to be *weakly invariant* if there exists a function $\sigma \to q(\sigma)$ such that for all σ, C_1, C_2, we have

$$d(\sigma C_1, \sigma C_2) = q(\sigma) d(C_1, C_2).$$

The distance is said to be *defined modulo* Σ (of up to Σ, or insensitive to Σ, etc.) if for all σ, C we have $d(\sigma C, C) = 0$. (Note that in the literature, the term "invariant" is often used for the last definition).

For example, the distance $d^{(3)}$ is invariant by rotation, and weakly scale invariant (with $q(\sigma) = \sqrt{\lambda}$ when σ is a scaling with factor λ). This distance, however, is not defined modulo these transformations. We now show how modifications can be made to obtain distances which see shapes modulo scaling and/or rotations.

3.1. Scale invariance. To obtain a scale-invariant expression for the infinitesimal energy (12), it is natural to normalize it by l. This yields

$$\delta E^{(2)}(\lambda, g, r) = (\lambda - 1)^2 + \int_0^1 \left[(\dot{g}_s - 1)^2 + |r - 1|^2\right] ds. \tag{23}$$

For this energy we need another kind of Hilbertian isometric identification of G. Define $\mathcal{L}_0^2$ to be the unit sphere of $\mathcal{L}^2$; that is,

$$\mathcal{L}_0^2 = \left\{ X, X \in \mathcal{L}^2, \int_0^1 |X|^2 ds = 1 \right\}.$$

Admissible paths in $\mathcal{L}_0^2$ are taken to be admissible paths in $\mathcal{L}^2$ which stay in $\mathcal{L}_0^2$, and we define $\Phi_0 : \mathcal{L}_0^2 \to G$ to be the restriction of Φ to $\mathcal{L}_0^2$. We also denote by G_0 the set of all (g, r) such that $(1, g, r) \in G$, and $\tilde{G}_0 = \mathcal{L}_0^2 \cap \tilde{G}$. We may, with a slight abuse of language, also consider Φ_0 as a map from $\mathcal{L}_0^2$ to G_0. We can also identify $\mathcal{L}^2$ to $]0, +\infty[\times \mathcal{L}_0^2$, G to $]0, +\infty[\times G_0$ and Φ to the map, between $]0, +\infty[\times \tilde{G}_0$ and $]0, +\infty[\times G_0$, which associates $(\lambda, \Phi_0(X))$ with a couple (λ, X).

The left-invariant distance on $]0, +\infty[$ such that the infinitesimal norm is $|\alpha|$ is $d(\lambda, \tilde{\lambda}) = |\log \lambda - \log \tilde{\lambda}|$. Moreover, the shortest paths in $\mathcal{L}_0^2$ are, like in finite dimension, the great circles (this may be proved by elementary arguments). The length of the shortest great circle linking X and Y is $\arccos \int_0^1 \Re(X(s).\overline{Y}(s))ds$. Like in the previous paragraph, minimizing this expression among all X and Y in $\mathcal{L}_0^2$ such that $\Phi(X) = a$ and $\Phi(Y) = b$ leads to the choice of X and Y such that $\Re(X\overline{Y}) \geq 0$ everywhere. An elementary computation provides the equation of the arc of great circle between X and Y:

$$t \to \frac{\sin[L(1-t)]}{\sin L} X + \frac{\sin Lt}{\sin L} Y\,,$$

with $\sin L > 0$ (L is the length of the path). It is easily seen that if $\Re(X\overline{Y}) \geq 0$, the above path stays in $\tilde{G}_0$ for all t. We thus have the following theorem.

THEOREM 2. *One defines a distance on G by (for $a = (\lambda, g, e^{i\Delta})$, $b = (\mu, h, e^{i\tilde{\Delta}})$)*

$$d_G^{(2)}(a, b) = \left[|\log \lambda - \log \mu|^2 + 4 \left(\arccos \int_0^1 \sqrt{\dot{g}_s \dot{h}_s} \left| \cos\left(\frac{\Delta - \tilde{\Delta}}{2}\right) \right| ds \right)^2 \right]^{1/2}. \tag{24}$$

One defines a scale-invariant distance between two plane curves C and $\tilde{C}$, with normalized angle functions θ and $\tilde{\theta}$, by putting

$$d^{(2)}(C, \tilde{C}) = \left\{ |\log l - \log \tilde{l}|^2 + 4 \left[\inf_g \arccos \int_0^1 \sqrt{\dot{g}_s(s)} \left| \cos \frac{\tilde{\theta} \circ g(s) - \theta(s)}{2} \right| ds \right]^2 \right\}^{1/2}, \tag{25}$$

the infimum being taken over functions g which are strictly increasing diffeomorphisms of $[0, 1]$.

The proof is straightforward once it has been noticed that $d_G^{(2)}$ is left-invariant with $\gamma(a) \equiv 1$.

To define a distance which is defined modulo scale, one simply needs to drop the term $|\log l - \log \tilde{l}|$ since the angle function θ characterizes a curve up to translation and scaling. Moreover, this choice is optimal (relatively to our distance). The best way to match two curves C and $\tilde{C}$ modulo the scale is to renormalize them so that their length is the same. We have the following theorem.

THEOREM 3. *One defines a distance (modulo translations and homotheties) between two plane curves C and $\tilde{C}$, with normalized angle functions θ and $\tilde{\theta}$, by putting*

$$d^{(1)}(C, \tilde{C}) = 2 \inf_g \arccos \int_0^1 \sqrt{\dot{g}_s(s)} \left| \cos \frac{\tilde{\theta} \circ g(s) - \theta(s)}{2} \right| ds, \tag{26}$$

the infimum being taken over functions g which are strictly increasing diffeomorphisms of $[0, 1]$.

3.2. Rotation invariance. The distances above are rotation invariant but not defined modulo rotation (the action of a rotation on a pair $(g,r) \in G_0$ being defined by $(g,r) \to (g, e^{ic}r)$ where c is the angle of rotation). However, because of this invariance, the quotient distances on G_0, which are

$$d_G^{(0)}(a,b) = \inf\{d_G^{(1)}(a, Rb), R \text{ plane rotation}\},$$

and on plane curves,

$$d^{(0)}(C, \tilde{C}) = \inf\{d^{(1)}(C, R\tilde{C}), R \text{ plane rotation}\},$$

are distances modulo similarities (rotations, scalings, and translations). Since rotations merely translate the angle functions, this yields the following theorem.

THEOREM 4. *One defines a distance on G_0 by (for $a = (g, e^{i\Delta})$, $b = (h, e^{i\tilde{\Delta}})$)*

$$(27) \qquad d_G^{(0)}(a,b) = 2 \min_{c \in]-\pi,\pi]} \left[\arccos \int_0^1 \sqrt{\dot{g}_s \dot{h}_s} \left| \cos\left(\frac{\Delta - \tilde{\Delta} - c}{2} \right) \right| ds \right] 1/2.$$

One defines a distance (modulo similarities) between two plane curves C and $\tilde{C}$, with normalized angle functions θ and $\tilde{\theta}$, by putting

$$(28) \qquad d^{(0)}(C, \tilde{C}) = 2 \inf_g \min_{c \in]-\pi,\pi]} \arccos \int_0^1 \sqrt{\dot{g}_s(s)} \left| \cos \frac{\tilde{\theta} \circ g(s) - \theta(s) - c}{2} \right| ds\,.$$

This distance may also be interpreted as the length of a shortest path. We consider again small displacements in the neighborhood of e_G but do it directly on $\mathcal{L}_0^2$ since G_0 and $\mathcal{L}_0^2$ are identified. The cost of an infinitesimal displacement $\mathbf{1} \to \mathbf{1} + \xi dt$ was $4dt^2 \int_0^1 |\xi|^2 ds$. Making an infinitesimal rotation e^{-icdt} yields the cost $4dt^2 \int_0^1 |\xi - ic|^2 ds$, and this is minimized by taking $c = \int_0^1 \Im(\xi) ds$. So, the cost of this infinitesimal translation is

$$\delta d(\mathbf{1}, \mathbf{1} + \xi dt)^2 = 4dt^2 \int_0^1 \left| \xi - i \int_0^1 \Im(\xi) ds \right|^2 ds\,.$$

By left-translation, the cost of an infinitesimal translation $X \to X + \eta dt$ must be

$$\delta d(X, X + \eta dt)^2 = \delta d \left(\mathbf{1}, \mathbf{1} + \frac{\eta \circ g_X^{-1}}{X \circ g_X^{-1}} dt \right)^2 = 4dt^2 \int_0^1 \left| \eta - iX \int_0^1 \Im(\eta) \overline{X} ds \right|^2 ds$$

after a change of variable. By construction, the length of a curve $\mathbf{X}(t,.)$ is the minimum, over all choices of $c(t)$, of

$$(29) \qquad 2 \int_0^1 \left[\int_0^1 |\dot{\mathbf{X}}_t - ic(t)\mathbf{X}|^2 ds \right]^{1/2} dt;$$

Letting $\mathbf{D}(t) = \int_0^t c(u) du$, however, this is precisely the length of the path $e^{-i\mathbf{D}(t)}\mathbf{X}(t,.)$, for the usual norm on $\mathcal{L}_0^2$. In other terms, within the family of admissible paths going from the class of $\mathbf{X}(0,.)$ modulo rotations to the class of $\mathbf{X}(1,.)$ modulo rotations, we can find one of which the length, as given by (29), is the same as its geodesic length on $\mathcal{L}_0^2$.

3.3. Remark. In [15], a study of 2-dimensional shapes is made. In this reference, a shape is the class of a *labeled* k-ad of points in the plane modulo similarities. The space of shapes with k vertices is then identified with $\mathbb{C}P^{k-2}$. Identifying the plane with $\mathbb{C}$, it is obtained that if $z = (z_1, \dots, z_k)$ and $w = (w_1, \dots, w_k)$ are two sequences of k points in $\mathbb{C}$, their distance, or more precisely, the distance of their class modulo, the action of similarities is

$$d_K(z, w) = \arccos\left(\left|\sum_{j=1}^{k} z_j^* \overline{w}_j^*\right|\right), \tag{30}$$

where z^* (and, similarly, w^*) is the centered and normalized version of z:

$$z_l^* = \frac{z_l - \sum_j z_j/k}{\sqrt{\sum_j |z_j|^2/k}}.$$

This is also equal to

$$d_K(z, w) = \arccos\left[\max_c \left(\sum_{j=1}^{k} \Re(z_j^* \overline{w}_j^* e^{-ic})\right)\right]. \tag{31}$$

This distance is, at least formally, very close to the distance $d^{(0)}$, which has just been computed, although the last one leaves the possibility to compare k-ads with k'-ads, for $k \neq k'$, and does not require (even if $k = k'$) that vertices are matched together. It is, however, instructive to look at what is obtained when $k = k'$ and vertices are constrained to be matched together (which can also make sense, for example, if some information is carried by the vertices). So, let z and w be k-ads as above, and define Z and W as the $k-1$-ads formed by the edges of z and w (i.e., $Z_j = z_{j+1} - z_j$); our distance is (the optimal matching is piecewise linear; cf. section 6)

$$\tilde{d}^{(0)}(Z, W) = \arccos\left[\max_c \left(\sum_{j=1}^{k-1} \left|\Re\left[\left(e^{-ic} Z_j^\star \overline{W}_j^\star\right)^{1/2}\right]\right|\right)\right], \tag{32}$$

where the normalization now is such that $\sum_j |Z_j^\star| = 1$.

So, in addition to the transformation $z \to Z$ (so that k-ads are represented by edges rather than vertices), we see that the main difference between the distances is the apparition of the square root in (32). In fact, we have

$$\tilde{d}^{(0)}(Z, W) = \inf d_K(U, V),$$

the minimum being searched among all U, V such that $U_j^2 = Z_j$ and $V_j^2 = W_j$.

Consider now that our k-ads are piecewise linear interpolations of two plane curves C and $\tilde{C}$ with length 1, and angle functions θ and $\tilde{\theta}$, such that, for some diffeomorphism $g : [0, 1] \to [0, 1]$, the arc-length parameters s_j of z_j and $\tilde{s}_j$ of w_j are such that $\tilde{s}_j = g(s_j)$. Making the approximations $Z_j = (s_j - s_{j-1})e^{i\theta(s_j)}$, we can rewrite (32) as

$$\tilde{d}^{(0)}(Z, W) = \arccos\left[\max_c \left(\int_0^1 \sqrt{\dot{g}_s} \left|\cos\left(\frac{\theta - \tilde{\theta} \circ g - c}{2}\right)\right| ds\right)\right], \tag{33}$$

which gives back $d^{(0)}$ if it is minimized with respect to g. Using (30), we get an alternative expression for $d^{(0)}$ which is

$$d^{(0)}(C,\tilde{C}) = \arccos\left[\sup_g \sup_\epsilon \left|\int_0^1 \epsilon(s)\sqrt{\dot{g}_s} e^{i\frac{\theta-\tilde{\theta}\circ g}{2}} ds\right|\right], \tag{34}$$

where ϵ is a measurable function such that $\epsilon(s) = \pm 1$.

4. Proofs of results. This paragraph provides the proof of two results which have been stated above (Proposition 5 and Lemma 1). It may be skipped without harming the understanding of the rest of the paper.

4.1. Proof of Proposition 5. We prove that *if two admissible paths in* $\mathcal{L}^2$, $\mathbf{X}(t,.)$, *and* $\mathbf{Y}(t,.)$ *satisfy* $\mathbf{X}(t,.)^2 = \mathbf{Y}(t,.)^2$ *for all* t, *then,* $\dot{\mathbf{X}}_t(t,.)^2 = \dot{\mathbf{Y}}_t(t,.)^2$.

We need a lemma.

LEMMA 2. *Let q be a differentiable map from $\mathbb{C}$ into $\mathbb{C}$, such that there exists a constant A such that $|q'(x)| \le A$ and for all h $|q(x+h) - q(x) - q'(x).h| = A|h|^2$. Then, if $\mathbf{X}(t,.)$ is admissible in $\mathcal{L}^2$, $q \circ \mathbf{X}(t,.)$ is also admissible, and its derivative is*

$$t \to q'(\mathbf{X}(t,.)).\dot{\mathbf{X}}_t(t,.),$$

where q' is the differential of q, identified with a 2×2 matrix.

Let us see how Lemma 2 can be used to prove Proposition 5. Let $K_n(u)$ be a smooth function on $\mathbb{R}^+$ such that $K_n(u) = 1$ for $0 \le u \le n$, $K_n(u) = 0$ for $u > n+1$, and $0 \le K_n(u) \le 1$ for all u. Set $q_n(x) = K_n(|x|^2).x^2$. The hypotheses of Lemma 2 are true for q_n. Moreover, since $X^2 = Y^2$, $q_n(\mathbf{X}) = q_n(\mathbf{Y})$, and we get, after differentiation, denoting by q_n' the derivative of q_n

$$q_n'(X).\dot{\mathbf{X}}_t = q_n'(Y).\dot{\mathbf{Y}}_t,$$

with $q_n'(x).z = 2K_n(|x|^2)zx + 2K_n'(|x|^2)x^2\Re(\overline{x}z)$. This yields

$$2\dot{\mathbf{X}}_t\mathbf{X}K_n(|\mathbf{X}|^2) + \mathbf{X}^2K_n'(|\mathbf{X}|^2)\Re(\bar{\mathbf{X}}\dot{\mathbf{X}}_t) = 2\dot{\mathbf{Y}}_t\mathbf{Y}K_n(|\mathbf{Y}|^2) + \mathbf{Y}^2K_n'(|\mathbf{Y}|^2)\Re(\bar{\mathbf{Y}}\dot{\mathbf{Y}}_t).$$

Since $K_{n+1}'K_n = 0$ and $K_nK_{n+1} = 1$, we get, multiplying by $K_n(|\mathbf{X}|^2) = K_n(|\mathbf{Y}|^2)$, the above equation at $n+1$,

$$2\dot{\mathbf{X}}_t\mathbf{X}K_n(|\mathbf{X}|^2) = 2\dot{\mathbf{Y}}_t\mathbf{Y}K_n(|\mathbf{Y}|^2).$$

Taking the squares and dividing by $X^2 = Y^2$, which are positive almost everywhere by assumption, we obtain that, for all n, $\dot{\mathbf{X}}_t^2(K_n(|\mathbf{X}|^2))^2 = \dot{\mathbf{Y}}_t^2(K_n(|\mathbf{Y}|^2))^2$ which implies that $\dot{\mathbf{X}}_t^2 = \dot{\mathbf{Y}}_t^2$.

We now prove Lemma 2. Note that, by hypothesis,

$$\int_0^1\int_0^1 \left|q'(\mathbf{X}(t,s))\dot{\mathbf{X}}_t(t,s)\right|^2 dt \le A^2\int_0^1\int_0^1 |\dot{\mathbf{X}}_t(t,s)|^2 dtds < \infty.$$

Fix $\phi \in \mathcal{L}^2$ and consider the mapping

$$\xi : t \to \int_0^1 q[\mathbf{X}(t,s)]\overline{\phi}(s)ds.$$

We must check that ξ has a generalized derivative given by

$$\dot{\xi}_t : t \to \int_0^1 q'(\mathbf{X}(t,s))\dot{\mathbf{X}}_t(t,s)\overline{\phi}(s)ds.$$

Let ψ be a C^∞ function, with compact support included in $]0,1[$. We want to show that

$$(35)\quad -\int_0^1\int_0^1 q[\mathbf{X}(t,s)]\overline{\phi}(s)\dot{\psi}_t(t)dsdt = \int_0^1\int_0^1 q'(\mathbf{X}(t,s))\dot{\mathbf{X}}_t(t,s)\overline{\phi}(s)\psi(t)dsdt\,.$$

Clearly, it is enough to prove this for bounded ϕ. Assume that the support of ψ is included in $[\delta, 1-\delta]$ for some $\delta > 0$. The left-hand term of (35) is the limit, when ϵ tends to 0, of

$$\int_0^1 \xi(t).(\psi(t-\epsilon)-\psi(t))/\epsilon dt\,,$$

which is equal (for $\epsilon < \delta$) to

$$\int_0^1 \psi(t).(\xi(t+\epsilon)-\xi(t))/\epsilon dt;$$

however, we have

$$\begin{aligned}\xi(t+\epsilon)-\xi(t) &= \int_0^1 \left[q[\mathbf{X}(t+\epsilon,s)] - q[\mathbf{X}(t,s)]\right]\overline{\phi}(s)ds\\ &= \int_0^1 q'[\mathbf{X}(t,s)].\left[\mathbf{X}(t+\epsilon,s)-\mathbf{X}(t,s)\right]\overline{\phi}(s)ds + \int_0^1 R(t,s,\epsilon)\overline{\phi}(s)ds,\end{aligned}$$

where $R(t,s,\epsilon)$ is such that

$$\int_0^1 |R(t,s,\epsilon)|\overline{\phi}(s)ds \le C\int_0^1 |\mathbf{X}(t+\epsilon,s)-\mathbf{X}(t,s)|^2 ds\,,$$

using the fact that ϕ and ψ are bounded. Yet, for almost all s, $\mathbf{X}(t+\epsilon,s)-\mathbf{X}(t,s) = \int_t^{t+\epsilon}\dot{\mathbf{X}}_t(u,s)du$; thus,

$$(1/\epsilon)\int_0^1\int_0^1 |R(t,s,\epsilon)|\overline{\phi}(s)dsdt \le C\int_0^1\int_0^1\int_t^{t+\epsilon} |\dot{\mathbf{X}}_t(u,s)|^2 dudtds,$$

which tends to 0 with ϵ. It remains, therefore, to show that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_0^1\int_0^1 \psi(t)q'[\mathbf{X}(t,s)].\left[\frac{\mathbf{X}(t+\epsilon,s)-\mathbf{X}(t,s)}{\epsilon}\right]\overline{\phi}(s)dsdt\\ &\quad = \int_0^1\int_0^1 \psi(t)q'[\mathbf{X}(t,s)].\dot{\mathbf{X}}_t(t,s)\overline{\phi}(s)dsdt.\end{aligned}$$

More generally, however, we have, for any f such that $f(t,s)=0$ when $t\le\delta$ or $t>1-\delta$,

$$(36)\quad \lim_{\epsilon\to 0}\int_0^1\int_0^1 f(t,s).\left[\frac{\mathbf{X}(t+\epsilon,s)-\mathbf{X}(t,s)}{\epsilon}\right]dsdt = \int_0^1\int_0^1 f(t,s)\dot{\mathbf{X}}_t(t,s)ds,$$

provided that $\int_0^1\int_0^1 |f(t,s)|^2 dsdt < \infty$. Indeed, still using the fact that

$$\begin{aligned}\int_\delta^{1-\delta} dt\int_0^1 \left|\frac{\mathbf{X}(t+\epsilon,s)-\mathbf{X}(t,s)}{\epsilon}\right|^2 ds &\le \int_\delta^{1-\delta} dt\int_0^1 \frac{1}{\epsilon}\int_\epsilon^{t+\epsilon} |\dot{\mathbf{X}}_t(u,s)|^2 duds\\ &\le \int_0^1\int_0^1 |\dot{\mathbf{X}}_t(u,s)|^2 duds\,,\end{aligned}$$

it suffices to show (36) for f in any dense subset of $\mathcal{L}^2$. This is true, by definition, for any finite linear combination of functions of the kind $\psi(t)\overline{\phi}(s)$, which form a dense subset.

4.2. Proof of Lemma 1. We now prove that $d^{(3)}$ is a distance. Note that one can write

$$[d^{(3)}(C,\tilde{C})]^2 = (\sqrt{l}-\sqrt{\tilde{l}})^2 + 2\sqrt{l\tilde{l}}\left(1-\sup_g \int_0^1 \sqrt{\dot{g}_s(s)}\left|\cos\frac{\tilde{\theta}\circ g(s)-\theta(s)}{2}\right| ds\right)$$

so that $d^{(3)}(C,\tilde{C}) = 0 \Rightarrow l = \tilde{l}$, and we must prove that if

$$\sup_g \int_0^1 \sqrt{\dot{g}_s(s)}\left|\cos\frac{\tilde{\theta}\circ g(s)-\theta(s)}{2}\right| ds = 1,$$

then $\theta = \tilde{\theta}$. So, let g^n be a sequence such that $g^n(s) = \int_0^s (q^n(\sigma))^2 d\sigma$, $g^n(1) = 1$ and

$$\int_0^1 q^n(s)\left|\cos\frac{\theta\circ g^n(s)-\tilde{\theta}(s)}{2}\right| ds \to 1\,.$$

Note that we have

$$\begin{aligned} &1-\int_0^1 q^n(s)\left|\cos\frac{\theta\circ g^n(s)-\tilde{\theta}(s)}{2}\right| ds \\ &= \frac{1}{2}\int_0^1 (q^n(s)-1)^2 ds + \int_0^1 q^n(s)\left(1-\left|\cos\frac{\theta\circ g^n(s)-\tilde{\theta}(s)}{2}\right|\right) ds; \end{aligned} \tag{37}$$

thus, q^n tends to 1 in L^2. Moreover, since $g^n(s) = \int_0^s (q^n(\tilde{s}))^2 d\tilde{s}$, g^n converges to Id uniformly on $[0,1]$. In addition, with a change of variables $s = g^n(u)$

$$\int_0^1 \left(\frac{1}{q^n\circ(g^n)^{-1}(s)}-1\right)^2 ds = \int_0^1 (q^n(u)-1)^2 du\,,$$

so that $1/q^n\circ(g^n)^{-1}$ also converges to $\mathbf{1}$.

By (37),

$$\int_0^1 q^n(s)\left(1-\left|\cos\frac{\theta\circ g^n(s)-\tilde{\theta}(s)}{2}\right|\right) ds$$

tends to 0. By Lusin's theorem, there exists, for all $\epsilon > 0$, a function θ^ϵ which is continuous on $[0,1]$ and equal to θ everywhere but on a set N^ϵ of Lebesgue measure smaller than ϵ. The quantity

$$\int_0^1 q^n(s)\left|\cos\frac{\theta^\epsilon\circ g^n(s)-\tilde{\theta}(s)}{2}\right| ds$$

tends to

$$\int_0^1 \left|\cos\frac{\theta^\epsilon(s)-\tilde{\theta}(s)}{2}\right| ds$$

when n tends to infinity; however,

$$\int_0^1 q^n(s)\left|\left|\cos\frac{\theta^\epsilon\circ g^n(s)-\tilde\theta(s)}{2}\right|-\left|\cos\frac{\theta\circ g^n(s)-\tilde\theta(s)}{2}\right|\right|ds$$

is smaller than

$$2\int_0^1 q^n(s)\mathbf{1}_{g^n(s)\in N^\epsilon}=2\int_0^1(1/q^n\circ(g^n)^{-1})\mathbf{1}_{N^\epsilon}(s)ds,$$

and this quantity tends to $2\int_0^1\mathbf{1}_{N^\epsilon}\le 2\epsilon$ when n tends to infinity. Similarly,

$$\int_0^1\left|\left|\cos\frac{\theta^\epsilon(s)-\tilde\theta(s)}{2}\right|-\left|\cos\frac{\theta(s)-\tilde\theta(s)}{2}\right|\right|ds\le 2\epsilon\,.$$

Thus,

$$\lim_n\int_0^1\left|q^n(s)\left|\cos\frac{\theta\circ g^n(s)-\tilde\theta(s)}{2}\right|-\left|\cos\frac{\theta(s)-\tilde\theta(s)}{2}\right|\right|ds\le 4\epsilon$$

for all $\epsilon>0$. This implies that $\int_0^1|\cos\frac{\theta(s)-\tilde\theta(s)}{2}|ds=1$, and this is possible only if $\theta(s)=\tilde\theta(s)$ (modulo 2π) for almost all s.

5. The case of closed curves. Our object set $\mathcal{C}$ consists of all sufficiently smooth plane curves, including also closed curves. This implies that the distances we have used is also valid for comparing closed curves, but some modification is required in order to obtain a pertinent comparison. Indeed, representing the curve by its length and its angle function implicitly implies that some starting point was fixed for the arc-length parametrization of our curves. In the case of open curves, there only are two such choices, and each of them should be tried to get the best match. For closed curves, the starting point may be anywhere, which complicates a little more the computational problem (while leaving it feasible). This can be reformulated within a more rigorous framework.

The subset $\mathcal{C}_c$ of closed curves in $\mathcal{C}$ is the set of all pairs (l,ζ) such that $\int_0^1\zeta(s)ds=0$. Unfortunately, the group G does not act on $\mathcal{C}_c$, since for no nontrivial $a\in G$ we have $a.\mathcal{C}_c\subset\mathcal{C}_c$, so that our previous construction cannot be applied to $\mathcal{C}_c$. A little can be done, however, by using the following remark: for any function $\zeta:[0,1]\to\mathbb{C}$, and any $u\in\mathbb{R}$, we can define $\tau_u.\zeta:[0,1]\to\mathbb{C}$ such that $\tau_u.\zeta(s)=\zeta(s+u)$ in which it is assumed that ζ has been expanded as a periodic function defined over all $\mathbb{R}$. This defines a new action on $\mathcal{C}$, and the main remark is that the distances we have defined are invariant with respect to this action. This implies that, d being any of the distances $d^{(3)}$ to $d^{(0)}$ above, one can define a distance on $\mathcal{C}$ modulo this last action by setting, for $C=(l,\zeta)$, $\tilde C=(\tilde l,\tilde\zeta)$

$$d_c(C,\tilde C)=\inf_u d(\tau_u C,\tilde C),\tag{38}$$

where $\tau_u C=(l,\tau_u\zeta)$.

The action of τ_u on C does not have any natural geometric meaning unless C is closed, in which case it simply corresponds to translating the origin of the arc-length parametrization without modifying the geometric curve associated to C. The distance

d_c is therefore our candidate for comparing closed curves. It should be noticed, however, that there is no constraint ensuring that during the optimal deformation process going from C to $\tilde{C}$ the curves remain closed. It is in fact authorized to "break" C before reaching $\tilde{C}$.

6. Numerical implementation.

6.1. Case of polygonal curves. Given two functions θ and $\tilde{\theta}$, the core of the numerical problem is to compute

$$\sup_g \int_0^1 \sqrt{\dot{g}_s(s)} \left|\cos \frac{\tilde{\theta} \circ g(s) - \theta(s)}{2}\right| ds\,.$$

This is not trivial since the functional to maximize is not concave, and even not differentiable because of the absolute value. The approach we have used is to approximate our curves by polygons, for which some explicit computation may be carried on, as we show now. So, assume that both curves are piecewise linear (i.e., that θ and $\tilde{\theta}$ are piecewise constant).

Thus, there exists $0 = s_0 < s_1 < \cdots < s_m = 1$ (resp. $0 = \tilde{s}_0 < \tilde{s}_1 < \cdots < \tilde{s}_n = 1$) and constants $\theta_1, \ldots, \theta_m$ (resp. $\tilde{\theta}_1, \ldots, \tilde{\theta}_n$) such that $\theta(s) \equiv \theta_i$ on $[s_{i-1}, s_i[$ (resp. $\tilde{\theta}(s) \equiv \tilde{\theta}_j$ on $[\tilde{s}_{j-1}, \tilde{s}_j[$). We have

$$\int_0^1 \sqrt{\dot{g}_s(s)} \left|\cos \frac{\tilde{\theta} \circ g(s) - \theta(s)}{2}\right| ds = \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} \sqrt{\dot{g}_s(s)} \left|\cos \frac{\tilde{\theta} \circ g(s) - \theta_i}{2}\right| ds.$$

We consider the points which match the s_i; that is, $\tau_i = g^{-1}(s_i)$, $i = 1, \ldots, m$. We begin by computing the best function g when the τ_i are fixed. Putting $h = g^{-1}$, the previous expression, which will be denoted by $Q(g)$, writes

$$Q(g) = \sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} \sqrt{\dot{h}_s(s)} \left|\cos \frac{\tilde{\theta}(s) - \theta_i}{2}\right| ds.$$

For $i = 0, \ldots, m$, let $j(i)$ be the index j for which $\tau_i \in [\tilde{s}_j, \tilde{s}_{j+1}[$. Set $k(i) = j(i+1) - j(i)$ and $\alpha_{ik} = s_{j(i)+k}$, for $k = 1, \ldots, k(i)$. Let also $\alpha_{i0} = \tau_i$ and $\alpha_{ik(i)+1} = \tau_{i+1}$. Finally, let $\beta_{ik} = \tilde{\theta}(\alpha_{ik}) = \tilde{\theta}_{j(i)+k+1}$ for $k = 0, \ldots, k(i)$. With these notations, we have

$$Q(g) = \sum_{i=1}^{m} \sum_{k=1}^{k(i)+1} \left|\cos \frac{\beta_{ik-1} - \theta_{i+1}}{2}\right| \int_{\alpha_{ik-1}}^{\alpha_{ik}} \sqrt{\dot{h}_s(s)} ds.$$

This implies that $Q(g)$ is maximal if h is linear on the intervals $[\alpha_{ik-1}, \alpha_{ik}[$, *which will be assumed from now.* Let $\gamma_{ik} = h(\alpha_{ik})$. Note that we must have $\gamma_{i0} = s_i$ and $\gamma_{ik(i)+1} = s_{i+1}$. We can write

$$Q(g) = \sum_{i=1}^{m} \sum_{k=1}^{k(i)+1} \left|\cos \frac{\beta_{ik-1} - \theta_{i+1}}{2}\right| \sqrt{\alpha_{ik} - \alpha_{ik-1}} \sqrt{\gamma_{ik} - \gamma_{ik-1}}$$

that is, putting $c_{ik} = \left|\cos \frac{\beta_{ik-1} - \theta_{i+1}}{2}\right| \sqrt{\alpha_{ik} - \alpha_{ik-1}}$ and $\delta_{ik} = \sqrt{\gamma_{ik} - \gamma_{ik-1}}$

$$Q(g) = \sum_{i=1}^{m} \sum_{k=1}^{k(i)+1} c_{ik} \delta_{ik}.$$

This expression may in turn be optimized with respect to the δ_{ik}, with the constraints $\delta_{ik} > 0$ and $\sum_{k=1}^{k(i)+1} \delta_{ik}^2 = s_{i+1} - s_i$. This gives

$$\delta_{ik} = \frac{c_{ik}\sqrt{s_{i+1} - s_i}}{\sqrt{\sum_{k=1}^{k(i)+1} c_{ik}^2}},$$

and, for these δ_{ik},

$$Q(g) = \sum_{i=1}^{m} \sqrt{(s_{i+1} - s_i) \sum_{k=1}^{k(i)+1} c_{ik}^2}\,. \tag{39}$$

This depends only on the τ_i. Returning to the initial notation, this is

$$Q(g) = \sum_{i=1}^{m} \sqrt{(s_{i+1} - s_i)Q_i}, \tag{40}$$

with

$$Q_i = \cos^2 \frac{\tilde{\theta}_{j(i)+1} - \theta_{i+1}}{2}(\tilde{s}_{j(i)+1} - \tau_i) + \sum_{j=j(i)+1}^{j(i+1)-1} \cos^2 \frac{\tilde{\theta}_{j+1} - \theta_{i+1}}{2}(\tilde{s}_{j+1} - \tilde{s}_j)$$
$$+ \cos^2 \frac{\tilde{\theta}_{j(i+1)} - \theta_{i+1}}{2}(\tau_{i+1} - s_{j(i+1)}).$$

We see that there exists a combinatorial part in the maximization of Q, which is due to the $j(i), i = 1, \dots, m$. Each $j(i)$ may take any value between 1 and n, with the constraint that $j(1) \leq j(2) \leq \cdots \leq j(m)$. If the $j(i)$ are fixed, the τ_i may be obtained by the maximization of a smooth function, with the constraint that for all i,

$$\max(\tilde{s}_{j(i)}, \tau_{i-1}) \leq \tau_i < \min(\tilde{s}_{j(i)+1}, \tau_{i+1})\,. \tag{41}$$

Now, $Q(g)$, as given in equation (40), may very quickly be maximized by linear programming, when the number of edges in the polygonal curve is not too large. When the number of edges is large, a suboptimal steepest-descent procedure may be used.

6.2. General curves. When one deals with general differentiable curves, each of them may be replaced by a polygonal approximation. We generally use a multiscale approach, starting with a rough polygonal approximation for which dynamic programming can be used, and then refine the result for enhanced approximations by steepest-descent.

To estimate the rotation parameter c in the expression of $d^{(0)}$, we start with an initial value c_0, find the optimal g with this fixed c_0, and then compute the best c given g. The procedure can be iterated a few times.

7. Experiments. We present examples from a small database composed with eight outlines of planes for four types of planes. The shapes have been extracted from 3-dimensional synthesis images under two slightly different view angles for each plane. We have applied some smooth stochastic noise to the outlines in order to obtain variants of the same shape which look more realistic. The outlines are presented in Figure 1. The lengths of the curves have been computed after smoothing (using a

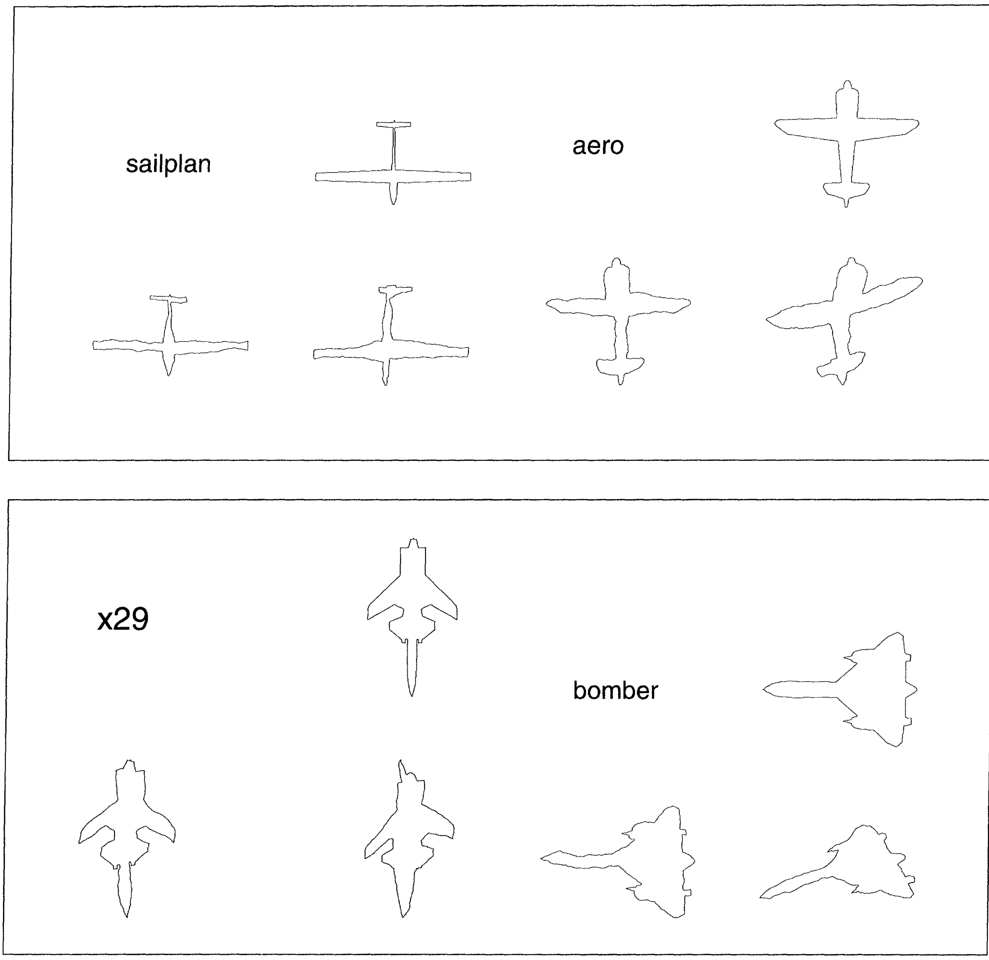

FIG. 1. *Outlines of planes from 4 classes. For each type of plane—upper right: original view (from above); lower left: view from above degraded by smooth noise; lower right: slight variation of the angle of view and noise. The compared outlines are lower left and lower right.*

TABLE 1
Matrix of distances (in radian) within the plane database.

	sailp-1	sailp-2	aero-1	aero-2	x29-1	x29-2	bomb-1	bomb-2
sailp-1	0	0.25	0.43	0.46	0.79	0.73	0.9	0.81
sailp-2	0.25	0	0.47	0.48	0.71	0.69	0.77	0.82
aero-1	0.43	0.47	0	0.28	0.76	0.8	0.77	0.81
aero-2	0.46	0.48	0.28	0	0.79	0.77	0.78	0.76
x29-1	0.79	0.71	0.76	0.79	0	0.38	0.84	0.81
x29-2	0.73	0.69	0.8	0.77	0.38	0	0.82	0.8
bomb-1	0.9	0.77	0.77	0.78	0.84	0.82	0	0.29
bomb-2	0.81	0.82	0.81	0.76	0.81	0.8	0.29	0

cubic-spline representation). The complete matrix of distances has been computed on this database and is given in Table 1. We see that the distance between a plane and the other one from the same class is always smaller than between any plane in another class. The computed distance is $d^{(0)}$ (insensitive to rotations), since the orientations of the planes vary.

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